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Vector Equilibrium Problems Under Asymptotic Analysis

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Abstract. Given a closed convex set K in $\mathbb{R}^n$; a vector function $F : K \times K \to \mathbb{R}^m$; a closed convex (not necessarily pointed) cone $P(x)$ in $\mathbb{R}^m$ with non-empty interior, int $P(x) \neq \emptyset$, various existence results to the problem

find $\bar{x} \in K$ such that $F(\bar{x}, y) \notin -\text{int } P(\bar{x}) \quad \forall\, y \in K$,

under $P(x)$-convexity/lower semicontinuity of $F(x, \cdot)$ and pseudomonotonicity on F, are established. Moreover, under a stronger pseudomonotonicity assumption on F (which reduces to the previous one in case $m = 1$), some characterizations of the non-emptiness of the solution set are given. Also, several alternative necessary and/or sufficient conditions for the solution set to be non-empty and compact are presented. However, the solution set fails to be convex in general. A sufficient condition to the solution set to be a singleton is also stated. The classical case $P(x) = \mathbb{R}^m_+$ is specially discussed by assuming semi-strict quasiconvexity. The results are then applied to vector variational inequalities and minimization problems. Our approach is based upon the computing of certain cones containing particular recession directions of K and F.

Key words: Convex vector optimization, Vector equilibrium problem, Vector variational inequalities, Scalar optimization, Weakly efficient solution, Efficient solution, Recession function, Recession cone, Convex analysis

1. Introduction

Given a closed convex set K in $\mathbb{R}^n$; a vector-valued function $F : K \times K \to \mathbb{R}^m$, the problem of determining the existence of $\bar{x} \in K$ such that

$$F(\bar{x}, x) \notin -\text{int } P(\bar{x}) \quad \forall\, x \in K, \tag{1.1}$$

has been the focus of attention of many mathematicians in recent years. The set-valued map $P : K \to \mathbb{R}^m$ is such that $P(x)$ is a non-empty closed convex (not necessarily pointed) cone with non-empty interior, int $P(x) \neq \emptyset$, for all $x \in K$, such a cone determines the underlying preference relation on $\mathbb{R}^m$.

The name of equilibrium problem, to our best knowledge, was coined in Blum and Oettli (1994) for the scalar version of the problem (1.1), i.e., $m = 1$. This kind of problems includes the classical ones in vector optimization, vector variational inequalities.

Scalar equilibrium problems ($m = 1$) were discussed in several papers, among others we quote Brezis et al. (1972), Joly and Mosco (1979), Blum and Oettli (1994), Bianchi and Schaible (1996), Hadjisavvas and Schaible (1998a), Hadjisavvas and Schaible (1998b), Bianchi et al. (1997), and more recently in Flores-Bazán (2000), Bianchi and Pini (2001); vector equilibrium problems as established in (1.1) with P being constant or not, were studied in Ansari (2000), Chadli and Riahi (2000), Tan and Tinh (1998), Oettli (1997), Bianchi et al. (1997), Hadjisavvas and Schaible (1998b), see also Hadjisavvas and Schaible (1998a), Giannessi (2000) for some survey on this subject.

The case $F(x, y) = G(y) - G(x)$ for some vector-valued function G with P constant has been dealt with in Chen and Craven (1994), Deng (1998a), Deng (1998b), Flores-Bazán (1999), and references therein, under the P-convexity assumption on F, and, in Ruíz-Canales and Rufían-Lizana (1995), Flores-Bazán (1999), under the (semi-strict) quasiconvexity condition. In Chen and Craven (1994) the differentiability together with $\mathbb{R}^m_+$-convexity of G are assumed. In this paper, such a problem is substituted, as in the scalar convex case, by the problem

$$\text{find } \bar{x} \in K \text{ such that } DG(\bar{x})(y - \bar{x}) \notin -\text{int}\,\mathbb{R}^m_+ \text{ for all } y \in K. \tag{1.2}$$

Here $DG(x)$ stands for the derivative of G at x, which is a matrix of order $m \times n$ given by the partial derivatives $\partial g_i/\partial x_j(x)$, where $G = (g_1, ..., g_m)$. Hence, the existence theorem proved in Chen and Craven (1994) will be derived as a consequence of our results (see Section 5). Related results can be also found in Siddiqi et al. (1995), Yang and Goh (1997).

In most of the preceding papers the boundedness of K is avoided. This is done, following the approach employed in Blum and Oettli (1994), Bianchi et al. (1997), Oettli (1997), by assuming the existence of a bounded set such that no element outside this set can be a candidate for solution. Therefore, in this case, the solution set will be bounded (Chen 1992; Chen and Craven 1994; Siddiqi et al. 1995; Daniilidis and Hadjisavvas 1996; Bianchi et al. 1997; Hadjisavvas and Schaible 1998a; Oettli 1997; Ansari 2000; Chadli and Riahi 2000). The results established in Tan and Tinh (1998), following the lines of Blum and Oettli (1994), allow the solution set be unbounded.

In this paper the non-compactness (unboundedness) assumption will be treated by using the notion of asymptotic functions and cones. This technique was also employed in Flores-Bazán (2000) for scalar equilibrium problems and in Deng (1998a,b), Flores-Bazán (1999) for vector minimization problems. Our results apply also to situations in which the solution set may be unbounded.

The purpose of the present paper is to provide necessary and/or sufficient conditions for the non-emptiness of the solution set to problem (1.1). In addition, if such solution set is required to be bounded, the necessary and/or sufficient condition become more precise.

In Section 2 we introduce some preliminary facts on asymptotic cones and vector-valued functions. Section 3 is devoted to establish the main existence theorems under certain conditions involving $P(x)$-convexity/lower semicontinuity on $F(x,\cdot)$ besides pseudomonotonicity. The sufficiency conditions will be also necessary under a stronger pseudomonotonicity assumption (see Theorems 3.11 and 3.14), which coincides with the previous one when $m = 1$. The classical case $P(x) = \mathbb{R}^m_+$, is studied extensively in Section 4 under the semi-strict quasiconvexity condition on each $f_i(x,\cdot)$, where $F = (f_1, ..., f_m)$. Applications to vector variational inequality and minimization problems are exhibited in Section 5.

The results of this paper are part of the Master Thesis of the second author carried out at the Departamento de Matemática de la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción.

2. Basic Facts and Preliminaries

Given any closed set K in $\mathbb{R}^n$ (actually the asymptotic notion to be considered is blind to closure), we define the asymptotic cone of K as the closed set

$$K^\infty = \Big\{x \in \mathbb{R}^n : \exists\, t_k \downarrow 0, \exists\, x_k \in K,\ \ t_k x_k \to x\Big\}.$$

We set $\emptyset^\infty = \emptyset$. For any given function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of h is defined as the function h^∞ such that

$$\text{epi}\, h^\infty = (\text{epi}\, h)^\infty.$$

Consequently, it is not difficult to prove that (Baiocchi et al., 1988)

$$h^\infty(y) = \inf\left[\liminf_{k\to+\infty} t_k h\left(\frac{x_k}{t_k}\right) : t_n \downarrow 0, x_k \to y\right].$$

In case K is also convex, it is known that for any given $x_0 \in K$,

$$K^\infty = \Big\{x \in \mathbb{R}^n : x_0 + tx \in K\ \forall\, t > 0\Big\}.$$

This cone is independent on x_0, and it gives rise to the notion of recession cone. Moreover, when h is a convex and l.s.c. function, we have

$$\begin{aligned} h^\infty(x) &= \lim_{\lambda\to+\infty} \frac{h(x_0+\lambda x) - h(x_0)}{\lambda} \\ &= \sup_{\lambda>0} \frac{h(x_0+\lambda x) - h(x_0)}{\lambda} \quad \forall\, x_0 \in \text{dom}\, h, \end{aligned}$$

where as usual, dom $h = \{x \in \mathbb{R}^n : h(x) < +\infty\}$. We notice the independence of h^∞ on the choice of x_0. In this case h^∞ is called the recession function of h. The epigraph of h is the set epi $h = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leqslant t\}$.

We collect some basic results on asymptotic cones in the next proposition that will be useful in the sequel.

PROPOSITION 2.1. *The following holds:*
(a) $K_1 \subset K_2$ *implies* $K_1^\infty \subset K_2^\infty$*;*
(b) $(K+x)^\infty = K^\infty$ *for all* $x \in \mathbb{R}^n$*;*
(c) $K^\infty = \{0\}$ *if, and only if,* K *is bounded;*
(d) *Let* (K_i), $i \in I$, *be any family of nonempty sets in* $\mathbb{R}^n$, *then*

$$\Big(\bigcap_{i\in I} K_i\Big)^\infty \subset \bigcap_{i\in I}(K_i)^\infty.$$

If, in addition, $\cap_i K_i \neq \emptyset$ *and each set* K_i, $i \in I$, *is closed and convex, then we obtain an equality in the previous inclusion.*

In what follows, P will denote a closed convex cone in $\mathbb{R}^m$, not necessarily pointed, such that $P \neq \mathbb{R}^m$.

DEFINITION 2.2. For a given closed convex cone $P \subset \mathbb{R}^m$, we say the mapping $H : K \to \mathbb{R}^m$:
(i) is P-convex if

$$\alpha H(y) + (1-\alpha)H(z) \in H(\alpha y + (1-\alpha)z) + P$$
$$\text{for all } \; y, z \in K, \text{ and all } \alpha \in [0, 1].]$$

(ii) is P-lower semicontinuous (P-lsc) at $x_0 \in K$ if for any neighborhood V of $H(x_0)$ in $\mathbb{R}^m$ there exists a neighborhood U of x_0 in $\mathbb{R}^n$ such that $H(U \cap K) \subset V + P$.
The mapping $H : K \to \mathbb{R}^m$ is said to be P-lsc in K if it is at every point $x_0 \in K$.

We point out that the $\mathbb{R}^m_+$-convexity/lower semicontinuity of H is equivalent to the (usual) convexity/lower semicontinuity of each component of H.

The next lemma can be also found in Tan and Tinh (1998) in the particular case $W = \mathbb{R}^m \setminus -\text{int}\, P$. It and the following proposition describe the geometrical interpretation of P-convexity and P-lower semicontinuity.

LEMMA 2.3. *Let* W *be a closed set in* $\mathbb{R}^m$ *such that* $W + P \subset W$*; let* $H : K \to \mathbb{R}^m$ *be* P*-lsc. Then, the set* $A = \{y \in K : \; H(y) \in -W\}$ *is closed. Therefore the set* $\{y \in K : \; H(y) \in -P\}$ *is also closed.*

Proof. Let (x_k), $k \in \mathbb{N}$, be any sequence in A such that $x_k \to \bar{x}$. We will prove that $\bar{x} \in A$. If, on the contrary $\bar{x} \notin A$, we could have $H(\bar{x}) \in \mathbb{R}^m \setminus -W$. Thus, by the P-lsc of H at $\bar{x}$, there is an open neighborhood U of $\bar{x}$ satisfying $H(U \cap K) \subset \mathbb{R}^m \setminus -W + P \subset \mathbb{R}^m \setminus -W$. Since $x_k \in U \cap K$ for k sufficiently large, the previous inclusion implies that $H(x_k) \in \mathbb{R}^m \setminus -W$ for k large enough, which contradicts the choice of $x_k \in A$, proving the desired result. □

Regarding the previous notions we have the following proposition. Part (b) can be found in Bianchi et al. (1997) and Part (a) follows from the very definition.

For additional relationship between these notions we refer to Luc (1989). In what follows

$$\text{epi } H = \Big\{(x, y) \in K \times \mathbb{R}^m : \ y \in H(x) + P\Big\}.$$

PROPOSITION 2.4. *Let $H : K \to \mathbb{R}^m$. Then,*
(a) epi H *is convex if and only if H is P-convex;*
(b) *Assume* int $P \neq \emptyset$, *H is P-lsc if and only if* $\{x \in K : \ H(x) - \lambda \notin \text{int } P\}$ *is closed for all $\lambda \in \mathbb{R}^m$.*

3. Main Existence Results

In this section we will devoted to the study of the problem

$$\text{find } \bar{x} \in K \text{ such that } \ F(\bar{x}, y) \notin -\text{int } P(\bar{x}) \ \text{ for all } y \in K. \tag{3.1}$$

The set of solutions to this problem is denoted by E_W. We also consider the problem

$$\text{find } \bar{x} \in K \text{ such that } \ F(y, \bar{x}) \notin \text{int } P(y) \ \text{ for all } y \in K, \tag{3.1$'$}$$

and its set of solutions is denoted by E'_W.

The following abstract result, which is not contained in any of the results existing in Oettli (1997), Bianchi et al. (1997), Hadjisavvas and Schaible (1998b), Tan and Tinh (1998), is proven by applying a similar reasoning as that used in the proof of Lemma 1 in Oettli (1997), and it cannot be obtained from any result for scalar equilibrium problem (as a device described in Oettli 1997) since we deal with a moving cone $P(x)$. A related result can be found in Chadli and Riahi (2000).

THEOREM 3.1. *Let K be a convex and compact set in $\mathbb{R}^n$. Let W be any set-valued mapping with non-empty values. Let $F : K \times K \to \mathbb{R}^m$ be a vector-valued mapping satisfying the following assumptions:*

($A0$) *$F(x, x) \in W(x)$ for all $x \in K$;*
($A1$) *for all $x, y \in K$, $F(x, y) \in W(x)$ implies $F(y, x) \in -W(y)$;*
($A2$) *for all $x \in K$, the set $\{\xi \in K : F(x, \xi) \in -W(x)\}$ is closed;*
($A3$) *for all $x \in K$, the set $\{\xi \in K : F(x, \xi) \notin W(x)\}$ is convex;*
($A4$) *for all $y \in K$: $F(x, y) \in -W(x)$ for all $x \in K$ implies $F(y, x) \in W(y)$ for all $x \in K$.*

Then, the solution set to the problem

$$\textit{find } \bar{x} \in K \textit{ such that } \ F(\bar{x}, y) \in W(\bar{x}) \ \textit{ for all } y \in K,$$

and that of the problem

$$\textit{find } \bar{x} \in K \textit{ such that } \ F(y, \bar{x}) \in -W(y) \ \textit{ for all } y \in K$$

are non-empty and both coincide, and they are closed.

Proof. We first find $\bar{x} \in K$ such that

$$\bar{x} \in \bigcap_{y \in K} \Big\{ x \in K : \ -F(y,x) \in W(y) \Big\}.$$

To that end, we shall use the famous KKM lemma (see for instance Aubin, 1979; Fan, 1961). Set

$$G(y) = \Big\{ x \in K : \ -F(y,x) \in W(y) \Big\}.$$

Because of assumption $(A2)$, each set $G(y)$ is closed and bounded. In order to apply the KKM lemma, we need to prove that $\mathrm{co}\{y_1, ..., y_k\} \subset \bigcup_i G(y_i)$ for every $k \in \mathbb{N}$. If $y = \sum_{i=1}^k \alpha_i y_i \notin \bigcup_{i=1}^k G(y_i)$ for some $\alpha_i \geqslant 0$, $i = 1, ..., k$, $\sum_{i=1}^k \alpha_i = 1$, then $y \notin G(y_i)$ for all $i = 1, ..., k$. Thus $-F(y_i, y) \notin W(y_i)$, which implies $F(y, y_i,) \notin W(y)$ by assumption $(A1)$. Thus $F(y, y) \notin W(y)$ because of assumption $(A3)$, which contradicts assumption $(A0)$. This proves that co $\{y_1, ..., y_k\} \subset \bigcup_{i=1}^k G(y_i)$ for all $k \in \mathbb{N}$. An application of the KKM lemma yields the existence of $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$, i.e., $-F(y, \bar{x}) \in W(y)$ for all $y \in K$, in other words, the second problem has solution. Now we apply assumption $(A4)$ to conclude that such a solution is also a solution of the first problem. Since every solution to the first is a solution to the second by assumption $(A1)$, we conclude that both solution sets coincide. The closedness is a consequence of $(A2)$. □

We now adapt the previous abstract result to our problem and we shall give simpler verifiable conditions on P and F ensuring the validity of all assumptions imposed in Theorem 3.1.
The basic assumptions on P are listed in the following hypothesis (H0), whereas the basic assumptions on F are listed in hypothesis (H1) below.

HYPOTHESIS (H0). The set-valued map $P : K \to \mathbb{R}^m$ is such that $P(x)$ is a non-empty closed convex (not necessarily pointed) cone with non-empty interior, int $P(x) \neq \emptyset$, for all $x \in K$;

HYPOTHESIS (H1). The vector-valued map $F : K \times K \to \mathbb{R}^m$ is such that
(f_0) for all $x \in K$, $F(x,x) \in l(P(x)) \doteq P(x) \cap (-P(x))$;
(f_1) for all $x, y \in K$, $F(x,y) \notin -\mathrm{int}\, P(x)$ implies $F(y,x) \notin \mathrm{int}\, P(y)$;
(f_2) for all $x \in K$, the mapping $F(x,\cdot) : K \to \mathbb{R}^m$ is $P(x)$-convex and $P(x)$-lsc.;
(f_3) for all $x, y \in K$, the set $\{\xi \in [x,y] : F(\xi, y) \notin -\mathrm{int}\, P(\xi)\}$ is closed. Here $[x,y]$ stands for the closed line segment joining x and y.

By using hypotheses (H0) and (H1), Theorem 3.1 yields the next result.

LEMMA 3.2. *Let K be a convex and compact set in $\mathbb{R}^n$. Let P be a set-valued map satisfying hypothesis* (H0). *Let $F : K \times K \to \mathbb{R}^m$ be a vector-valued mapping satisfying hypothesis* (H1). *Then $E_W = E'_W$ is a non-empty closed set, i.e., there exists $\bar{x} \in K$ such that $F(\bar{x}, x) \notin -\mathrm{int}\, P(\bar{x})$ for all $x \in K$.*

Proof. We only need to verify the assumptions $(A2)$, $(A3)$ and $(A4)$ of Theorem 3.1 for $W(x) = \mathbb{R}^m \setminus -\text{int } P(x)$. Assumption $(A2)$ follows by applying Lemma 2.3 to $F(x, \cdot)$ and $W(x)$ as before; assumption $(A3)$ is a direct consequence of (H0), the $P(y)$-convexity of $F(y, \cdot)$ and the fact $\mathbb{R}^m \setminus -W(y) \subset P(y)$ for all $y \in K$. Let us verify assumption $(A4)$: take any $y \in K$ such that $F(x, y) \notin \text{int } P(x)$ for all $x \in K$. For every $x \in K$ consider $x_t = y + t(x - y)$ for $t \in \;]0, 1[$. Clearly $x_t \in K$. The $P(x_t)$-convexity of $F(x_t, \cdot)$ implies

$$tF(x_t, x) + (1 - t)F(x_t, y) \in F(x_t, x_t) + P(x_t).$$

Since $F(x_t, y) \notin \text{int} P(x_t)$, from the previous inclusion one has

$$\begin{aligned} tF(x_t, x) &\in P(x_t) + (1 - t)(\mathbb{R}^m \setminus -\text{int } P(x_t)) \subset \\ &\subset P(x_t) + (\mathbb{R}^m \setminus -\text{int } P(x_t)) \subset \mathbb{R}^m \setminus -\text{int } P(x_t). \end{aligned}$$

It follows that $F(x_t, x) \notin -\text{int } P(x_t)$. Letting $t \downarrow 0$, we obtain by assumption (f_3), $F(y, x) \notin -\text{int } P(y)$. Since x was arbitrary, the desired result is proved. □

REMARK 3.3.

(*i*) We notice from the last part of the proof of the previous lemma (see the verification of assumption $(A4)$) that under hypothesis (H0) and assumptions (f_0), (f_2) and (f_3) of hypothesis (H1), we have actually proved that

$$F(y, \bar{x}) \notin \text{int } P(y) \quad \forall\, y \in K \;\text{ implies }\; F(\bar{x}, y) \notin -\text{int } P(\bar{x}) \quad \forall\, y \in K.$$

This implication is related to a certain maximal pseudomonotonicity condition already discussed in Oettli (1997).

(*ii*) In the proof of the preceding lemma only the assumption $F(x, x) \in P(x)$ for all $x \in K$ has been used. The condition $F(x, x) \in -P(x)$ will be needed when we deal with K unbounded.

(*iii*) In the case when P is non-constant, condition (f_3) of hypothesis (H1) is fulfilled, if, for all $y \in K$, the mapping $F(\cdot, y)$ is continuous (in the usual sense) along line segments in K and the set-valued map $\mathbb{R}^m \setminus -\text{int } P(x)$ has closed graph. If on the contrary, P is independent of x, condition (f_3) is satisfied if, for all $x, y \in K$, the mapping $G : [0, 1] \to \mathbb{R}^m$ defined by $G(t) = F(ty + (1-t)x, y)$, is P-upper semicontinuous, in the sense that, for all $t_0 \in [0, 1]$ and any neighborhood V of $G(t_0)$ in $\mathbb{R}^m$ there exists $\delta \in \;]0, 1[$ such that $G(t) \subset V - P$ for all $t \in \;]t_0 - \delta, t_0 + \delta[\cap[0, 1]$.

REMARK 3.4. Looking carefully at the proof of Lemma 3.2 one can realize that the same result holds under assumptions (f_0') and (f_1') instead of (f_0) and (f_1) in hypothesis (H1), where

(f_0') for all $x \in K$, $F(x, x) \in l(W(x)) \doteq W(x) \cap (-W(x))$, here $W(x) = \mathbb{R}^m \setminus -\text{int } P(x)$;

(f_1') for all $x, y \in K$, $F(x, y) \notin -\text{int } P(x)$ implies $F(y, x) \in -P(y)$.

Certainly assumption (f_0') is weaker than (f_0) whereas (f_1') is stronger than (f_1).

Let us consider, just for a moment, the problem of finding

$$\bar{x} \in K \text{ such that } F(\bar{x}, y) \in P(\bar{x}) \text{ for all } y \in K. \tag{3.2}$$

We denote its solution set by E_P. Let us see that Theorem 3.1 will imply also an existence result to problem (3.2) provided $P(x) \cup (-P(x)) = \mathbb{R}^m$ for all $x \in K$. The latter condition together with hypothesis (H0) say that $P(x)$ has not to be pointed, indeed, $P(x)$ must be a closed half-space. Thus problem (3.2) has a more precise formulation. The conclusion is summarized in the following corollary. If P is constant, a related result may be found in Oettli (1997), Corollary 4 (compare assumptions (i) and (v) of Corollary 2 in Oettli 1997). Simplified optimality conditions in case $F(x, y) = h(y) - h(x)$ with P being a constant pointed closed convex cone is discussed in Flores-Bazán and Oettli (2001).

COROLLARY 3.5. *Let K be a closed convex and bounded set in $\mathbb{R}^n$. Let P be a set-valued map satisfying hypothesis* (H0). *Let $F : K \times K \to \mathbb{R}^m$ be a vector-valued map such that assumptions (f_1) and (f_3) of hypothesis* (H1) *are verified with $P(x)$ instead of $\mathbb{R}^m \setminus -\text{int } P(x)$. Assume, in addition, that $P(x) \cup (-P(x)) = \mathbb{R}^m$ for all $x \in K$. Then E_P is a non-empty closed set, i.e., there exists $\bar{x} \in K$ such that $F(\bar{x}, x) \in P(\bar{x})$ for all $x \in K$.*

In order to deal with the unbounded case, i.e., when K is an unbounded set, it is necessary to describe the asymptotic behavior of F along some particular directions. These directions are determined by the following cones

$$R_0 = \bigcap_{y \in K} \Big\{ v \in K^\infty : F(y, z + \lambda v) \notin \text{int } P(y) \ \forall \lambda > 0, \\ \forall z \in K \text{ such that } F(y, z) \in -P(y) \Big\},$$

$$R_1 = \bigcap_{y \in K} \Big\{ v \in K^\infty : F(y, y + \lambda v) \notin \text{int } P(y) \ \forall \lambda > 0 \Big\},$$

which are nonempty (because of assumption (f_0)) closed cones not necessarily convex. Clearly $R_0 \subset R_1$. Scalar versions of these sets have been introduced in Flores-Bazán (2000) and extend those defined in Flores-Bazán (1999). Remark that R_1 may be computed in an easier way than R_0.
The importance of such cones lies in the following two results.

PROPOSITION 3.6. *Let K be a closed convex set in $\mathbb{R}^n$. Assume that P satisfies hypothesis* (H0). *Let $F : K \times K \to \mathbb{R}^m$ such that $F(x, \cdot) : K \to \mathbb{R}^m$ is $P(x)$-*

convex and satisfies $F(x,x) \in P(x)$ *for all* $x \in K$. *Then*

$$\bigcap_{y\in K} \left\{v \in K^\infty :\ F(y, y+\lambda v) \notin \operatorname{int} P(y)\ \ \forall\, \lambda > 0\right\} \subset$$
$$\subset \bigcap_{y\in K} \left\{v \in K^\infty :\ F(y+\lambda v, y) \notin -\operatorname{int} P(y+\lambda v)\ \ \forall\, \lambda > 0\right\}.$$

Proof. Set $W(x) = \mathbb{R}^m \setminus -\operatorname{int} P(x)$. Let $v \in K^\infty$ be in the set of the left hand-side of the previous inclusion. Then for any $y \in K$ and $\lambda > 0$, the $P(y+\lambda v)$-convexity of $F(y+\lambda v, \cdot)$ implies

$$\frac{1}{2}F(y+\lambda v, y+\lambda v+\lambda v) + \frac{1}{2}F(y+\lambda v, y)$$
$$\in F(y+\lambda v, y+\lambda v) + P(y+\lambda v).$$

Then

$$\begin{aligned}\frac{1}{2}F(y+\lambda v, y) &\in P(y+\lambda v) + \frac{1}{2}W(y+\lambda v) + P(y+\lambda v) \subset \\ &\subset P(y+\lambda v) + W(y+\lambda v) \subset W(y+\lambda v).\end{aligned}$$

Thus $F(y+\lambda v, y) \in W(y+\lambda v)$. Since $y \in K$ and $\lambda > 0$ were arbitrary, we conclude the proof. □

Remark that, if F also satisfies assumption (f_1) then we have the equality in the previous proposition.

THEOREM 3.7. *Let* K *be a closed convex set, let* P *satisfies hypothesis* (H0). *Assume the mapping* $F : K \times K \to \mathbb{R}^m$ *satisfies hypothesis* (H1). *Then*

$$(E_W)^\infty \subset R_0 \subset R_1 \subset \bigcap_{y\in K} \left\{x \in K : F(x,y) \notin -\operatorname{int} P(x)\right\}^\infty \subset$$
$$\subset \bigcap_{y\in K} \left\{x \in K : F(y,x) \notin \operatorname{int} P(y)\right\}^\infty.$$

If, in addition, there exists $\bar{x} \in K$ *such that* $F(y,\bar{x}) \in -P(y)$ *for all* $y \in K$, *then* $R_0 = (E_W)^\infty$.

Proof. Set $W(x) = \mathbb{R}^m \setminus -\operatorname{int} P(x)$, $x \in K$. Let us prove the first inclusion. Let $v \in (E_W)^\infty$, then there exist $t_k \downarrow 0$, $u_k \in E_W$ such that $t_k u_k \to v$. For $y \in K$ arbitrary, we have $F(u_k, y) \in W(u_k)$ for all $k \in \mathbb{N}$. In addition, take any $z \in K$ such that $F(y,z) \in -P(y)$. Let us fix any $\lambda > 0$. For k sufficiently large, the $P(y)$-convexity of $F(y,\cdot)$ implies

$$(1-\lambda t_k)F(y,z) + \lambda t_k F(y,u_k) \in F(y, (1-\lambda t_k)z + \lambda t_k u_k) + P(y).$$

Hence

$$-F(y,(1-\lambda t_k)z+\lambda t_k u_k) \in W(y)+P(y) \subset W(y).$$

From Lemma 2.3, it follows that $F(y, z+\lambda v) \in -W(y)$. This proves $v \in R_0$.

The second inclusion is straightforward and the proof of the third inclusion is as follows. Let $v \in K^{\infty}$ such that $-F(y, y+\lambda v) \in W(y)$ for all $\lambda > 0$ and all $y \in K$. By the previous proposition $F(y+\lambda v, y) \in W(y+\lambda v)$ for all $\lambda > 0$ and all $y \in K$. For any fixed $y \in K$, set $x_k \doteq y+kv \in K$, $k \in \mathbb{N}$. Then $F(x_k, y) \in W(x_k)$ for all $k \in \mathbb{N}$. By choosing $t_k = 1/k$, we have $t_k x_k = y/k + v \to v$ as $k \to +\infty$, i.e., $v \in \{x \in K : F(x, y) \in W(x)\}^{\infty}$. Since y was arbitrary, the proof of the third inclusion is complete. The fourth inclusion is a consequence of assumption (f_1).

Let us prove the last part of the theorem. First observe that $F(y, \bar{x}) \in -P(y) \subset -W(y)$ for all $y \in K$ implies that $F(\bar{x}, y) \in W(\bar{x})$ for all $y \in K$ by Remark 3.3 Thus $\bar{x} \in E_W$. Let $v \in R_0$, we have, in particular, $F(y, \bar{x}+\lambda v) \in -W(y)$ for all $\lambda > 0$ for all $y \in K$. It turns out that $F(\bar{x}+\lambda v, y) \in W(\bar{x}+\lambda v)$ for all $\lambda > 0$ and all $y \in K$ by Remark 3.3 again, showing that $\bar{x}+\lambda v \in E_W$ for all $\lambda > 0$. Hence $v \in (E_W)^{\infty}$. □

Concrete applications of Theorem 3.7 and the following one will be given in Section 5. For example, when dealing with convex vector minimization problems like in Chen and Craven (1994), we can recover from the previous theorem and our main result the existence theorem established in Chen and Craven (1994) and part of Deng (1998a) as well.

We recall the following assumption introduced in Remark 3.4

(f_1') for all $x, y \in K$, $F(x, y) \notin -\text{int}\, P(x)$ implies $F(y, x) \in -P(y)$,

which is a reinforcement of (f_1). A class of vector functions satisfying (f_1') in $\mathbb{R}^2$ for $P(x) = \mathbb{R}^2_+$ is given by

$$F(x, y) = (f_1(x, y), f_2(\|x-y\|)),$$

with f_1 being any pseudomonotone function ($f_1(x, y) \geqslant 0 \Longrightarrow f_1(y, x) \leqslant 0$) and $f_2(t) \leqslant 0$ for all $t \geqslant 0$.

We consider also the sharper closed cone

$$R_0' = \bigcap_{y \in K} \Big\{ v \in K^{\infty} : F(y, z+\lambda v) \notin \text{int}\, P(y) \ \ \forall\, \lambda > 0, \\ \forall\, z \in K \text{ such that } F(y, z) \notin \text{int}\, P(y) \Big\}.$$

Obviously $R_0' \subset R_0 \subset R_1$. The proof of the next theorem follows the lines of the previous one.

THEOREM 3.8. *Let K be a closed convex set, let P satisfies hypothesis* (H0). *Assume the mapping $F : K \times K \to \mathbb{R}^m$ satisfies hypothesis* (H1) *with* (f_1') *instead* (f_1). *Then*

$$(E_W)^\infty \subset R_0' \subset R_0 \subset R_1 \subset \bigcap_{y\in K} \Big\{x \in K : F(x, y) \notin -\text{int } P(x)\Big\}^\infty \subset$$

$$\subset \bigcap_{y\in K} \Big\{x \in K : F(y, x) \in -P(y)\Big\}^\infty.$$

If, in addition, $E_W \neq \emptyset$ *then* $R_0' \subset (E_W)^\infty$. *As a consequence* $R_0' = (E_W)^\infty$.

In case we are interested in problem (3.2) above, Theorem 3.7 or Theorem 3.8 yields the following corollary. We recall that E_P denotes the set of solutions to problem (3.2).

COROLLARY 3.9. *Let K be a closed convex set, let P satisfies hypothesis* (H0). *Under assumptions of Corollary* 3.5, *if* $E_P \neq \emptyset$, *then*

$$(E_P)^\infty = \bigcap_{y\in K} \Big\{ v \in K^\infty : F(y, z + \lambda v) \in -P(y) \ \forall \lambda > 0,$$

$$\forall z \in K \text{ such that } F(y, z) \in -P(y)\Big\}.$$

Proof. Let $\bar{x} \in E_P$. By assumption (f_1) we have $F(y, \bar{x}) \in -P(y)$ for all $y \in K$. Thus we can apply the preceding theorem or Theorem 3.7 to conclude with the desired result. □

We are now in a position to establish our first main existence theorem.

THEOREM 3.10. *Let K be a closed convex set in $\mathbb{R}^n$; let P satisfies hypothesis* (H0). *Moreover, let $F : K \times K \to \mathbb{R}^m$ be a vector-valued map satisfying hypothesis* (H1). *If, in addition F is such that*

(∗) *for every sequence (x_k) in K, $\|x_k\| \to +\infty$ such that $x_k/\|x_k\| \to v$ with $v \in R_0$ (or $v \in R_1$) and for all $y \in K$ it exists k_y such that $F(x_k, y) \notin -\text{int } P(x_k)$ for all $k \geqslant k_y$, there exists $u \in K$ such that $\|u\| < \|x_k\|$ and $F(x_k, u) \in -P(x_k)$ for $k \in \mathbb{N}$ sufficiently large,*

then problem (3.1) *admits a solution and the solution set, E_W, is closed.*

Proof. For every $k \in \mathbb{N}$, set $K_k = \{x \in K : \|x\| \leqslant k\}$. We may suppose, without loss of generality, that $K_k \neq \emptyset$ for all $k \in \mathbb{N}$. Let us consider the problem

$$\text{find } \bar{x} \in K_k \text{ such that } F(\bar{x}, y) \notin -\text{int } P(\bar{x}) \ \forall y \in K_k. \tag{3.3}$$

By Lemma 3.2, problem (3.3) admits a solution, say $x_k \in K_k$ for all $k \in \mathbb{N}$. If $\|x_k\| < k$ for some $k \in \mathbb{N}$, then, we claim that x_k is also a solution to problem (3.1). In fact, if there is $y \in K$ with $\|y\| > k$ such that $F(x_k, y) \in -\text{int } P(x_k)$, we take $z \in K$ with $z \in \,]x_k, y[$ and $\|z\| < k$. Writing $z = \alpha x_k + (1-\alpha)y$ for some $\alpha \in \,]0, 1[$, we have by the $P(x_k)$-convexity of $F(x_k, \cdot)$,

$$\alpha F(x_k, x_k) + (1-\alpha)F(x_k, y) \in F(x_k, z) + P(x_k).$$

This implies

$$-F(x_k, z) \in -(1-\alpha)F(x_k, y) + P(x_k).$$

It follows that $-F(x_k, z) \in \operatorname{int} P(x_k)$, which contradicts the choice of x_k. Consequently x_k is a solution to the original problem.

We consider now the case $\|x_k\| = k$ for all $k \in \mathbb{N}$. We may suppose, without loss of generality, that $x_k/\|x_k\| \to v$, $v \neq 0$. Then $v \in K^\infty$. For any fixed $y \in K$ and $\lambda > 0$, we have $F(x_k, y) \notin -\operatorname{int} P(x_k)$ for all $k \in \mathbb{N}$ sufficiently large. In addition, take any $z \in K$ such that $F(y, z) \in -P(y)$. For k sufficiently large, the $P(y)$-convexity of $F(y, \cdot)$ implies

$$\begin{aligned}&\left(1 - \frac{\lambda}{\|x_k\|}\right) F(y, z) + \frac{\lambda}{\|x_k\|} F(y, x_k)\\ &\quad \in F\left(y, \left(1 - \frac{\lambda}{\|x_k\|}\right) z + \frac{\lambda}{\|x_k\|} x_k\right) + P(y).\end{aligned}$$

Hence

$$-F\left(y, \left(1 - \frac{\lambda}{\|x_k\|}\right) z + \frac{\lambda}{\|x_k\|} x_k\right) \notin -\operatorname{int} P(y).$$

Thus, by the $P(y)$-lsc of $F(y, \cdot)$ (see Lemma 2.3), $F(y, z + \lambda v) \notin \operatorname{int} P(y)$. This proves $v \in R_0$. By assumption, there exist $u \in K$ such that $\|u\| < \|x_k\|$ and $F(x_k, u) \in -P(x_k)$ for k sufficiently large. We claim that x_k is also a solution to problem (3.1). If not, then there exists $y \in K$, $\|y\| > k$ such that $F(x_k, y) \in -\operatorname{int} P(x_k)$. Since $\|u\| < \|x_k\|$ we can find $z \in \,]u, y[$ such that $\|z\| < k$. From the convexity,

$$\alpha F(x_k, u) + (1-\alpha)F(x_k, y) \in F(x_k, z) + P(x_k)$$

for some $\alpha \in \,]0, 1[$. Hence,

$$\begin{aligned}-F(x_k, z) &\in -\alpha F(x_k, u) - (1-\alpha)F(x_k, y) + P(x_k)\\ &\in P(x_k) + \operatorname{int} P(x_k) + P(x_k) \subset \operatorname{int} P(x_k).\end{aligned}$$

This contradicts the choice of x_k, proving that x_k is a solution to (3.1). □

We have point out that condition $(*)$ (respectively condition $(**)$ in the next theorem) holds vacuously in case $R_0 = \{0\}$ (respectively $R_0' = \{0\}$) and it is implied, for instance, by assuming $R_1 = \{0\}$ according to Theorem 3.7. Thus the sufficient conditions imposed in Chen and Craven (1994), Deng (1998a) are recovered.

In a similar way we also obtain

THEOREM 3.11. *Let K be a closed convex set in $\mathbb{R}^n$; let P satisfies hypothesis* (H0). *Moreover, let $F : K \times K \to \mathbb{R}^m$ be a vector-valued map satisfying hypothesis* (H1) *with* (f_1') *instead* (f_1). *Then problem* (3.1) *admits a non-empty closed solution set if and only if property* $(**)$ *is satisfied, where*

$(**)$ *for every sequence (x_k) in K, $\|x_k\| \to +\infty$ such that $x_k/\|x_k\| \to v$ with $v \in R_0'$ and for all $y \in K$ it exists k_y such that $F(x_k, y) \notin -\text{int}\, P(x_k)$ for all $k \geqslant k_y$, there exists $u \in K$ such that $\|u\| < \|x_k\|$ and $F(x_k, u) \in -P(x_k)$ for $k \in \mathbb{N}$ sufficiently large.*

Proof. The "if" part is similar to that of the previous theorem. The "only if" part is obtained as follows. Take any sequence (x_k) in K such that $\|x_k\| \to +\infty$ and any $\bar{x} \in E_W$. Then, by virtue of assumption (f_1'), condition $(**)$ is satisfied by setting $u = \bar{x}$ and choosing x_k with k sufficiently large such that $\|x_k\| > \|\bar{x}\|$. □

REMARK 3.12.

(i) A stronger assumption than (f_1') and therefore of (f_1) is the following
   (f_1'') for all $x, y \in K$, $F(x, y) \notin -\text{int}\, P(x)$ implies $F(y, x) \in -\text{int}\, P(y)$.
   Such a condition implies E_W is a singleton as one can deduce immediately.

(ii) A condition implying $(*)$, is the extension to the vector case of the so-called Karamardian's condition (Karamardian 1971) widely used in the study of complementarity problems:
   $(+)$ there exists a non-empty compact set $D \subset K$ such that $\forall\, x \in K \setminus D$, $\exists\, y \in D$: $F(x, y) \in -P(x)$.

We point out that condition $(+)$ in contrast to condition $(++)$ below, applies to situations in which the solution set to problem (3.1) may be unbounded. Notice that the cone R_0 (or (R_1)) is not mentioned explicitely in $(+)$.

Sometimes one could be interested, maybe by numerical aspects, in knowing a priori, when the solution set is bounded. In this context, next theorems play important roles. Characterizations of the non-emptiness and boundedness of the solution set to scalar equilibrium problems were derived in Flores-Bazán (2000), which generalize those obtained in Daniilidis and Hadjisavvas (1999).

THEOREM 3.13. *Let K be a closed convex set in $\mathbb{R}^n$; let P be a set-valued map satisfying hypothesis* (H0). *Assume the vector-valued map $F : K \times K \to \mathbb{R}^m$ satisfies hypothesis* (H1). *Then,*

(a) *$R_0 = \{0\}$ implies the solution set, E_W, is non-empty and compact. The same conclusion is obtained if instead is assumed the condition*

$$\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r :\ F(x, y) \in -\text{int}\, P(x), \tag{3.4}$$

where $K_r = \{x \in K : \|x\| \leqslant r\} \neq \emptyset$;

(b) *under the additional assumption that there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \in -P(y)$ for all $y \in K$, we have: E_W is non-empty and compact $\Longleftrightarrow R_0 = \{0\} \Longleftrightarrow$ (3.4) holds.*

Proof. (a) We take the sequence (x_k) constructed in the proof of the previous theorem. We claim that such a sequence is bounded under either assumption $R_0 = \{0\}$ or (3.4). In fact, if not, up to a subsequence, we have $\|x_k\| \to +\infty$ and $x_k/\|x_k\| \to v$, thus $v \in K^\infty$ and $v \neq 0$. We argue exactly as in Theorem 3.10 to conclude that $v \in R_0$, which cannot happen if $R_0 = \{0\}$. On the other hand, $x_k \in K \setminus K_r$ for $k \in \mathbb{N}$ sufficiently large and $k > r$. Under assumption (3.4), there is $y_k \in K_r \subset K_k$ such that $F(x_k, y_k) \in -\text{int } P(x_k)$, contradicting the choice of x_k, proving the claim in case (3.4) is satisfied. Therefore, up to a subsequence, $x_n \to \bar{x}$, $\bar{x} \in K$. For any fixed $y \in K$, the choice of x_k implies $F(x_k, y) \notin -\text{int } P(x_k)$ for all k sufficiently large ($k > \|y\|$). Thus also $-F(y, x_k) \notin -\text{int } P(y)$ for all k sufficiently large. Applying Lemma 2.3, we obtain $F(y, \bar{x}) \notin \text{int } P(y)$ for all $y \in K$, proving that $\bar{x}$ is a solution to problem (3.1) by virtue of Remark 3.3. A similar reasoning proves also the boundedness of E_W under (3.4), and in the first case, it follows from Theorem 3.7. The closedness of E_W is as before.

(b) Assume first $E_W \neq \emptyset$ and bounded. We shall prove the coercivity condition (3.4) holds. If it was not so, in particular, for $k > \sup_{x \in E_W} \|x\| + 1$, there exists $x \in K \setminus K_k$ such that for all $y \in K_k$ one has $F(x, y) \notin -\text{int } P(x)$. Take $\lambda \in \,]0, 1[$ such that, setting $z = \bar{x} + \lambda(x - \bar{x})$, we have $k - 1 \leqslant \|z\| < k$. We claim that z is a solution to problem (3.3). In fact, for all $y \in K_k$, $\lambda F(y, x) + (1 - \lambda) F(y, \bar{x}) \in F(y, z) + P(y)$. Thus,

$$-F(y, z) \in P(y) + \lambda(\mathbb{R}^m \setminus -\text{int } P(y)) + (1 - \lambda) P(y) \subset \mathbb{R}^m \setminus -\text{int } P(y). \tag{3.5}$$

This together with Remark 3.3 imply that $F(z, y) \notin -\text{int } P(z)$ for all $y \in K_k$, proving that z is a solution to problem (3.3). We now prove that z is actually a solution to (3.1). If not, there exists $y \in K \setminus K_k$ satisfying $F(z, y) \in -\text{int } P(z)$. Choose $\tilde{z} = \alpha y + (1 - \alpha) z$ with $\alpha \in \,]0, 1[$ such that $\tilde{z} \in K_k$. Again, by the $P(z)$-convexity of $F(z, \cdot)$, $\alpha F(z, y) + (1 - \alpha) F(z, z) \in F(z, \tilde{z}) + P(z)$. This implies

$$\alpha F(z, y) \in (1 - \alpha) P(z) + (\mathbb{R}^m \setminus -\text{int } P(z)) + P(z) \subset \mathbb{R}^m \setminus -\text{int } P(z).$$

Hence $F(z, y) \notin -\text{int } P(z)$ which is a contradiction. The latter proves $z \in E_W$. On the other hand, by construction $\|z\| \geqslant k - 1 > \sup_{x \in E_W} \|x\| \geqslant \|z\|$, which cannot happen, proving the coercivity condition must hold. Because of Part (a), it only remains to prove that the boundedness of E_W implies $R_0 = \{0\}$. This is a consequence of Theorem 3.7, concluding the proof of the theorem. □

Analogously, we obtain the following characterization of the non-emptiness and boundedness of the solution set E_W under the more restrictive assumption (f_1').

THEOREM 3.14. *Let K be a closed convex set in $\mathbb{R}^n$; let P satisfies hypothesis* (H0). *Assume a vector-valued map $F : K \times K \to \mathbb{R}^m$ is given and satisfies hypothesis* (H1) *with (f_1') instead (f_1). Then, the following assertions are equivalent.*

(a) *E_W is non-empty and compact;*
(b) *$R'_0 = \{0\}$;*
(c) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r :\ F(x, y) \in -\text{int}\, P(x)$, *where* $K_r = \{x \in K :\ \|x\| \leqslant r\} \neq \emptyset$.

Proof. $(a) \iff (b)$: One implication follows from Theorem 3.11 since $(**)$ holds vacuosly if $R'_0 = \{0\}$. The other one is a consequence of Theorem 3.8.

$(a) \iff (c)$: These two implications are similar to the corresponding part in the preceding theorem. □

In many papers (e.g., Bianchi et al. 1997; Oettli 1997; Hadjisavvas and Schaible 1998a among others) the following condition was imposed:
$(++)$ there exist a nonempty compact subset $D \subset K$ and $y_0 \in D$ such that $F(x, y_0) \in -\text{int}\, P(x)$ for all $x \in K \setminus D$.
Such a condition has its origin in Brezis et al. (1972) and implies the solution set is bounded. More precisely, we shall prove that assumption $(++)$ together with hypotheses (H0) and (H1) imply $R_1 = \{0\}$. In fact, take any $v \in R_1$, $v \neq 0$. Then, in particular, $y_0 + \lambda v \in K \setminus D$ for all $\lambda > 0$ sufficiently large. Using assumption $(++)$ we get $F(y_0 + \lambda v, y_0) \in -\text{int}\, P(y_0 + \lambda v)$ for all $\lambda > 0$ sufficiently large. On the other hand, $v \in R_1$ implies $F(y, y + \lambda v) \notin \text{int}\, P(y)$ for all $\lambda > 0$ and all $y \in K$. By Proposition 3.6 we have in particular, $F(y_0 + \lambda v, y_0) \notin -\text{int}\, P(y_0 + \lambda v)$ for all $\lambda > 0$ sufficiently large, which contradicts a previous assertion. Hence we obtain the following result whose proof follows the previous reasoning and Theorems 3.7 and 3.10.

COROLLARY 3.15. *Under hypotheses* (H0) *and* (H1) *together with assumption* $(++)$, *the solution set to problem* (3.1) *is non-empty compact.*

REMARK 3.16. We close this section by mentioning that concerning problem 3.2 one can deduce also existence results by taking into account Corollary 3.5 and assuming additionally $P(x) \cup (-P(x)) = \mathbb{R}^m$ for all $x \in K$.

4. The Classical Case

In this section, we deal with the case $P(x) = \mathbb{R}^m_+$ $(m > 1)$, where $\mathbb{R}^m_+$ is the non-negative orthant in $\mathbb{R}^m$, and the components of $F(x, \cdot)$ are semi-strictly quasiconvex. Under the convexity condition all the results of this section follow from previous one. Thus, we are concerned with the problem

$$\text{find } \bar{x} \in K \text{ such that } F(\bar{x}, y) \notin -\text{int}\, \mathbb{R}^m_+ \ \ \forall\, y \in K. \tag{4.1}$$

The set of $\bar{x} \in K$ satisfying (4.1), is denoted by E_W. Setting $F(x, y) = (f_1(x, y), \ldots, f_m(x, y))$, an equivalent formulation of (4.1) is

$$\text{find } \bar{x} \in K \ \text{ such that } \ \forall\, y \in K,\ \exists\, i_y :\ f_{i_y}(\bar{x}, y) \geqslant 0.$$

In this case (see the previous section)

$$R_1 = \bigcap_{y\in K}\bigcup_{i=1}^{m}\Big\{v \in K^\infty :\ f_i(y, y+\lambda v) \leqslant 0\ \ \forall\, \lambda > 0\Big\}.$$

We denote by E_i the set of equilibrium points of f_i, that is, the set of $\bar{x} \in K$ such that $f_i(\bar{x}, y) \geqslant 0$ for all $y \in K$. It is clear that $E_i \subset E_W$.

Let us now recall some definitions.

DEFINITION 4.1. A function $f : K \to \mathbb{R}$ with K being a convex set,

(i) is said to be semi-strictly quasiconvex, if given any u, v in K, $f(u) \neq f(v)$, one has $f(z) < \max\{f(u), f(v)\}$ for all $z \in\ ]u, v[$;

(ii) is said to be quasiconvex if each of its level set is a convex set, or equivalently, if $f(tx + (1-t)y) \leqslant \max\{f(x), f(y)\}$ for all x, y in K and all $t \in [0, 1]$.

Originally a semi-strictly quasiconvex function was termed strictly quasiconvex, see for instance the monograph by Avriel et al. (1988). Simple examples show that there are functions that are semi-strictly quasiconvex but not quasiconvex. However, it is well known that any lsc and semi-strictly quasiconvex function is quasiconvex.

THEOREM 4.2. *Let K be a closed convex set. Assume that $f_i(x, x) = 0$ for all $x \in K$; for all $y \in K$, $f_i(y, \cdot)$, $i = 1, \ldots, m$, is lsc and quasiconvex. Furthermore, suppose that for all $x, y \in K$, $F(x, y) \notin -\mathrm{int}\,\mathbb{R}^m_+$ implies $F(y, x) \notin \mathrm{int}\,\mathbb{R}^m_+$. Then*

(a) $(E_W)^\infty \subset R_1$;

(b) *if, in addition, $E_i \neq \emptyset$ for all $i = 1, \ldots, m$ and*

$$E_i = \Big\{x \in K :\ f_i(y, x) \leqslant 0 \quad \forall\, y \in K\Big\}, \tag{4.2}$$

then

$$\bigcup_{i=1}^{m}\bigcap_{y\in K}\Big\{v \in K^\infty :\ f_i(y, y+\lambda v) \leqslant 0\ \ \forall\, \lambda > 0\Big\} \subset (E_W)^\infty.$$

Proof. Part (a). Take any $x_k \in E_W$ and $t_k \downarrow 0$ such that $t_k x_k \to v$. Let us fix any $y \in K$, then $F(x_k, y) \notin -\mathrm{int}\,\mathbb{R}^m_+$. Thus $F(y, x_k) \notin \mathrm{int}\,\mathbb{R}^m_+$, therefore there exists $i_k \in \{1, \ldots, m\}$ such that $f_{i_k}(y, x_k) \leqslant 0$. Since the set $\{1, \ldots, m\}$ is compact, we may assume that $i_0 \doteq i_{k_0} = i_k$ for all $k \geqslant n_0$. Hence

$$f_{i_0}(y, x_k) \leqslant 0 \quad \forall\, k \geqslant k_0.$$

By the quasiconvexity of $f_{i_0}(y, \cdot)$, we have for all $\lambda > 0$,

$$f_{i_0}(y, (1-\lambda t_k)y + \lambda t_k x_k) \leqslant \max\{f_{i_0}(y, y), f_{i_0}(y, x_k)\} = 0$$

for $k \in \mathbb{N}$ sufficiently large. The lower semicontinuity of $f_{i_0}(y, \cdot)$ implies $f_{i_0}(y, y + \lambda v) \leqslant 0$ for all $\lambda > 0$. This proves that $v \in R_1$.

Let us prove Part (b). It is proved in Theorem 3.3 of Flores-Bazán (2000) that

$$(E_i)^\infty = \bigcap_{y\in K} \Big\{ v \in K^\infty : \ f_i(y, y+\lambda v) \leqslant 0 \ \ \forall\, \lambda > 0 \Big\}.$$

Since $E_i \subset E_W$ for all $i = 1, ..., m$, the conclusion follows. □

Conditions under which (4.2) holds are given in Flores-Bazán (2000).

EXAMPLE 4.3.

(i) The following example shows, on one hand, the inclusions in Part (a) of the previous theorem may be strict, and on the other, Part (b) fails to be true if $E_i = \emptyset$ for some i. Take $K = \mathbb{R}^2$ and $F((x_1, x_2), (y_1, y_2)) = (\sqrt{|y_1|} - \sqrt{|x_1|}, e^{y_2} - e^{x_2})$. Then

$$R_1 = \Big(\{0\} \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \ ]-\infty, 0]\Big),$$

while $E_W = \{0\} \times \mathbb{R}$. Notice that $E_2 = \emptyset$.

(ii) Let $K = \mathbb{R}$, $F(x, y) = (\sqrt{|y|} - \sqrt{|x|}, y/(1+|y|) - x/(1+|x|))$. It is not difficult to show that $E_W = \ ]-\infty, 0] = R_1$.

In what follows, given a function $f : K \times K \to \mathbb{R}$, f^∞ will denote the recession function of $f(x, \cdot)$ for all $x \in K$, where we have extended $f(x, \cdot)$ by setting $f(x, y) = +\infty$ if $y \in \mathbb{R}^n \setminus K$.

COROLLARY 4.4. *Assume that for all* $x \in K$ $f_i(x, x) = 0$*; for all* $y \in K$ $f_i(y, \cdot)$, $i = 1, ..., m$, *is convex and lsc. Furthermore, suppose that for all* $x, y \in K$, $F(x, y) \notin -\text{int}\, \mathbb{R}^m_+$ *implies* $F(y, x) \notin \text{int}\, \mathbb{R}^m_+$. *Then,*

(a) *if* $E_W \neq \emptyset$, *we have*

$$(E_W)^\infty \subset \bigcap_{y\in K} \bigcup_{i=1}^{m} \Big\{ v \in K^\infty : \ f_i^\infty(y, v) \leqslant 0 \Big\};$$

(b) *if* $E_i \neq \emptyset$ *for all* $i = 1, ..., m$ *and*

$$E_i = \Big\{ x \in K : \ f_i(y, x) \leqslant 0 \ \ \forall\, y \in K \Big\},$$

then

$$\bigcup_{i=1}^{m} \Big\{ v \in K^\infty : \ f_i^\infty(y, v) \leqslant 0 \ \forall\, y \in K \Big\} \subset (E_W)^\infty. \tag{4.3}$$

Proof. This follows directly from the definition of recession function for convex functions. □

REMARK 4.5. An interesting situation occurs when all the sets $\{v \in K^\infty : f_i^\infty(y, v) \leqslant 0\}$ are independent of y (for instance if $f_i(x, y) = h_i(y) - h_i(x)$). In this case we have an equality in (4.3), for details we refer to Flores-Bazán (1999).

EXAMPLE 4.6.

(i) We show that the inclusions in Part (a) of the previous corollary may be strict. Take $K = \mathbb{R}^2$, $F((x_1, x_2), (y_1, y_2)) = (y_1^2 - x_1^2, e^{y_2} - e^{x_2})$. Then $f_1^\infty((y_1, y_2), (v_1, v_2)) = 0$ if $v_1 = 0$, $f_1^\infty((y_1, y_2), (v_1, v_2)) = +\infty$ elsewhere; $f_2^\infty((y_1, y_2), (v_1, v_2)) = 0$ if $v_2 \leqslant 0$, $f_2^\infty((y_1, y_2), (v_1, v_2)) = +\infty$ elsewhere. Thus,

$$R_1 = \Big(\{0\} \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \ ]-\infty, 0]\Big),$$

while $E_W = \{0\} \times \mathbb{R} = (E_W)^\infty$. Notice that $E_2 = \emptyset$.

(ii) Let $K = \mathbb{R}^2$ and $F((x_1, x_2), (y_1, y_2)) = (y_1^2 - x_1^2, y_2^2 - x_2^2)$. Then $f_i^\infty((y_1, y_2), (v_1, v_2)) = 0$ if $v_i = 0$, $f_i^\infty((y_1, y_2), (v_1, v_2)) = +\infty$ elsewhere. Thus,

$$R_1 = \Big(\{0\} \times \mathbb{R}\Big) \cup \Big(\mathbb{R} \times \{0\}\Big) = E_W.$$

Notice that E_W is not convex even if $F(x, \cdot)$ is $\mathbb{R}^m_+$-convex.

LEMMA 4.7. *Let K be a closed set; for $i = 1, \ldots, m$, assume that $f_i : K \times K \to \mathbb{R}$ is such that for all $x \in K$, $f_i(x, \cdot)$ is lsc Then, for all $x \in K$, the set $A = \{y \in K : F(x, y) \notin \operatorname{int} \mathbb{R}^m_+\}$ is closed.*

Proof. It follows from Lemma 2.3 since $\mathbb{R}^m$-lower semicontinuity of $F(x, \cdot)$ is equivalent to the usual lower semicontinuity of each $f_i(x, \cdot)$. □

LEMMA 4.8. *Let K be a convex compact set; assume that for $i = 1, \ldots, m$, $f_i : K \times K \to \mathbb{R}$ is such that for all $x \in K$, $f_i(x, \cdot)$ is lsc and semi-strictly quasiconvex, and $f_i(x, x) = 0$. In addition, suppose that for all $x, y \in K$, $F(x, y) \notin -\operatorname{int} \mathbb{R}^m_+$ implies $F(y, x) \notin \operatorname{int} \mathbb{R}^m_+$. Then, there exists $\bar{x} \in K$ such that $F(y, \bar{x}) \notin \operatorname{int} \mathbb{R}^m_+$ for all $y \in K$.*

Proof. Set

$$G(y) = \Big\{x \in K : \ F(y, x) \notin \operatorname{int} \mathbb{R}^m_+\Big\}, \ \ y \in K.$$

This is a closed set by the preceding lemma, and since K is bounded, it is compact. We will show that the convex hull $\operatorname{co}\{y_1, \ldots, y_k\} \subset \bigcup_i G(y_i)$ for every $k \in \mathbb{N}$ and then the conclusion will be a consequence of the KKM lemma as in Section 3. If $z = \sum_{i=1}^k \alpha_i y_i \notin \bigcup_{i=1}^k G(y_i)$ for some $\alpha_i \geqslant 0$, $i = 1, \ldots, k$, $\sum_{i=1}^k \alpha_i = 1$, then

$z \notin G(y_i)$ for all $i = 1, ..., k$. Thus $F(y_i, z) \in \text{int}\,\mathbb{R}^m_+$, which implies $F(z, y_i) \in -\text{int}\,\mathbb{R}^m_+$ for all $i = 1, ..., k$. Hence

$$f_j(z, y_i) < 0 \;\; \forall\, j = 1, ..., m, \;\; \forall\, i = 1, ..., k.$$

The quasiconvexity of f_j implies that $f_j(z, z) < 0$ for all $j = 1, ..., m$, which contradicts the assumption $F(x, x) = 0$, proving $\text{co}\{y_1, ..., k\} \subset \bigcup_{i=1}^k G(y_i)$ for all $k \in \mathbb{N}$. An application of the KKM lemma provides the existence of $\bar{x} \in K$ such that $\bar{x} \in \bigcap_{y \in K} G(y)$, which is the required solution. □

In order to solve problem (4.1), we shall require the following hypothesis

HYPOTHESIS (H2). *The vector-valued mapping* $F = (f_1, ..., f_m) : K \times K \to \mathbb{R}^m$ *is such that*

(f_0') *for all* $x \in K$, $F(x, x) = 0$;
(f_1') *for all* $x, y \in K$, $F(x, y) \notin -\text{int}\,\mathbb{R}^m_+$ *implies* $F(y, x) \notin \text{int}\,\mathbb{R}^m_+$;
(f_2') *for all* $x \in K$ *and all* $i = 1, ..., m$, *the function* $f_i(x, \cdot) : K \to \mathbb{R}$ *is semistricly quasiconvex and lsc;*
(f_3') *for all* $x, y \in K$, *the set* $\{\xi \in [x, y] : F(\xi, y) \notin -\text{int}\,\mathbb{R}^m_+\}$ *is closed.*

We remark that assumption (f_1') is weaker than requiring pseudomonotonicity in the sense of Karamardian for each f_i, that is,

$$\forall\, x, y \in K, \quad f_i(x, y) \geqslant 0 \Longrightarrow f_i(y, x) \leqslant 0.$$

REMARK 4.9. Lemma 4.8 asserts the existence of $\bar{x} \in K$ such that $F(y, \bar{x}) \notin \text{int}\,\mathbb{R}^m_+$ for all $y \in K$. We shall prove that under hypothesis (H2), essentially assumption (f_3'), one has $F(\bar{x}, y) \notin -\text{int}\,\mathbb{R}^m_+$ for all $y \in K$. In fact, let $y \in K$ and take $x_t = \bar{x} + t(y - \bar{x})$, $t \in\,]0, 1[$. Clearly $x_t \in K$. By assumption $f_j(x_t, \bar{x}) \leqslant 0$ for some $j \in \{1, ..., m\}$. We claim that $F(x_t, y) \notin -\text{int}\,\mathbb{R}^m_+$ for all $t \in\,]0, 1[$. If, on the contrary, there is $t \in\,]0, 1[$ such that $F(x_t, y) \in -\text{int}\,\mathbb{R}^m_+$, we have, by the semi-strict quasiconvexity in case $f_j(x_t, \bar{x}) = 0 > f_j(x_t, y)$,

$$0 = f_j(x_t, x_t) < \max\{f_j(x_t, \bar{x}), f_j(x_t, y)\} = 0,$$

which cannot happen. In the case $f_j(x_t, \bar{x}) < 0$, the quasiconvexity implies

$$0 = f_j(x_t, x_t) \leqslant \max\{f_j(x_t, \bar{x}), f_j(x_t, y)\} < 0,$$

which is also absurd. Hence $F(x_t, y) \notin -\text{int}\,\mathbb{R}^m_+$ for all $t \in\,]0, 1[$. Assumption (f_3') yields the desired result.

The following theorem is the first main result of this section.

THEOREM 4.10. *Assume that the vector function* $F = (f_1, ..., f_m) : K \times K \to \mathbb{R}^m$ *satisfies hypothesis* (H2). *Then, the solution set to problem* (4.1) *is non-empty and closed if the following property is satisfied:*

(∗) *for every sequence* (x_k) *in* K, $\|x_k\| \to +\infty$ *such that* $x_k/\|x_k\| \to v$ *with* $v \in R_1$ *and for all* $y \in K$ *it exists* k_y *such that* $F(x_k, y) \notin -\text{int}\,\mathbb{R}^m_+$ *for all* $k \geqslant k_y$, *there exists* $u \in K$ *such that* $\|u\| < \|x_k\|$ *and* $F(x_k, u) \in -\mathbb{R}^m_+$ *for* $k \in \mathbb{N}$ *sufficiently large.*

Proof. We proceed as in the proof of Theorem 3.10. For every $k \in \mathbb{N}$, set $K_k = \{x \in K : \|x\| \leqslant k\}$. We may suppose, without loss of generality, that $K_k \neq \emptyset$ for all $k \in \mathbb{N}$. Let us consider the problem

$$\text{find } \bar{x} \in K_k \text{ such that } F(\bar{x}, y) \notin -\text{int}\,\mathbb{R}^m_+ \;\; \forall\, y \in K_k. \tag{4.4}$$

By Lemma 4.8 and Remark 4.9, problem (4.4) admits a solution, say $x_k \in K_k$ for all $k \in \mathbb{N}$. If $\|x_k\| < k$ for some $k \in \mathbb{N}$, we claim that x_k is also a solution to problem (4.1). In fact, if not, there exists $y \in K$, $\|y\| > k$ such that $f_i(x_k, y) < 0$ for all $i = 1, ..., m$. Take any $z \in \,]x_k, y[$ such that $\|z\| < k$. By the choice of x_k, there exists i_k such that $f_{i_k}(x_k, z) \geqslant 0$. By the semi-strict quasiconvexity

$$f_{i_k}(x_k, z) < \max\{f_{i_k}(x_k, x_k), f_{i_k}(x_k, y)\} = f_{i_k}(x_k, x_k) = 0,$$

which contradicts the choice of i_k. Hence the claim is proved. Now, we assume that $\|x_k\| = k$ for all $k \in \mathbb{N}$. We may also suppose, without loss of generality, that $x_k/\|x_k\| \to v$. Then $v \in K^\infty$. For any fixed $y \in K$ and all $k > \|y\|$ there is i_k such that $f_{i_k}(x_k, y) \geqslant 0$. Using an argument similar to that used in Theorem 4.2, we conclude that $v \in R_1$. By property (∗), there exists $u \in K$, such that $\|u\| < \|x_k\|$ and $F(x_k, u) \in -\mathbb{R}^m_+$ for k sufficiently large. We claim that x_k is also a solution to problem (4.1). If not, there exists $y \in K$, $\|y\| > k$ such that $f_i(x_k, y) < 0$ for all $i = 1, ..., m$. Since $\|u\| < \|x_k\| = k$, we can find $z \in \,]u, y[$ such that $\|z\| < k$. Take $j \in \{1, ..., m\}$ such that $f_j(x_k, z) \geqslant 0$. We distinguish two cases, when $f_j(x_k, u) = 0 > f_j(x_k, y)$, one obtains by the semi-strict quasiconvexity,

$$0 \leqslant f_j(x_k, z) < f_j(x_k, u) = 0$$

which cannot happen. If $f_j(x_k, u) < 0$, then, by quasiconvexity

$$f_j(x_k, z) \leqslant \max\{f_j(x_k, u), f_j(x_k, y)\} < 0$$

contradicting the choice of j. Consequently x_k is a solution to problem (4.1). □

We know, from Theorem 4.5 in Flores-Bazán (2000), that

$$\bigcap_{y\in K} \left\{v \in K^\infty : f_i(y, y + \lambda v) \leqslant 0 \;\; \forall\, \lambda > 0\right\} = \{0\}$$
$$\Longleftrightarrow \quad E_i \text{ is a nonempty and compact set.}$$

Therefore, if $R_1 = \{0\}$, from Theorem 4.10 we have that E_W is non-empty and by Theorem 4.2, it is also compact. On the other hand, $R_1 = \{0\}$ implies

$$\bigcap_{y\in K} \left\{v \in K^\infty : \; f_i(y, y + \lambda v) \leqslant 0 \;\; \forall\, \lambda > 0\right\} = \{0\} \quad \text{for } i = 1, ..., m.$$

Thus, $E_i \neq \emptyset$ and compact for all $i = 1, ..., m$.

5. Vector Variational Inequalities and Minimization Problems

As before K is a convex closed set in $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ be a mapping taking values in the space of real $m \times n$ matrices, denoted by $\mathbb{R}^{m\times n}$. Let P be a set-valued map satisfying hypothesis (H0). Throughout this section $W(x) = \mathbb{R}^m \setminus -\text{int } P(x)$ or in case P is constant, $W = \mathbb{R}^m \setminus -\text{int } P$. In addition, we are also given a vector-valued function $h : \mathbb{R}^n \to \mathbb{R}^n$.

Let us consider the vector-valued mapping F defined as

$$F(x, y) = T(x)(y - h(x)), \; x, y \in K. \tag{5.1}$$

Assume that such a vector mapping satisfies hypothesis (H1) of Section 3. It is requested to

$$\text{find } \bar{x} \in K \text{ such that } T(\bar{x})(y - h(\bar{x})) \in W(\bar{x}) \quad \text{for all } y \in K. \tag{5.2}$$

The associated cone R_1 (see Section 3) is

$$R_1 = \bigcap_{y\in K} \Big\{ v \in K^\infty : \; T(y)(y + \lambda v - h(y)) \in -W(y), \; \forall\, \lambda > 0 \Big\}.$$

It follows that

$$R_1 \subset \Big\{ v \in K^\infty : \; T(y)(v) \in -W(y) \quad \forall\, y \in K \Big\} \subset R_0.$$

The first inclusion is trivial because of $F(y, y) = T(y)(y - h(y)) \in P(y) \cap (-P(y))$ for all $y \in K$ and $W(y) + P(y) \subset W(y)$. This implies, in the case when $P(x)$ is pointed for all $x \in K$, problem (5.2) is nothing else that the problem

$$\text{find } \bar{x} \in K \text{ such that } T(\bar{x})(y - \bar{x}) \in W(\bar{x}) \quad \text{for all } y \in K, \tag{5.3}$$

since $T(x)(y - h(x)) = T(x)(y - x) + T(x)(x - h(x)) = T(x)(y - x)$. The second inclusion is straightforward. Therefore, since $R_0 \subset R_1$, we have

$$R_0 = R_1 = \Big\{ v \in K^\infty : \; T(y)(v) \in -W(y) \quad \forall\, y \in K \Big\}. \tag{5.4}$$

Thus by Theorem 3.7

$$(E_W)^\infty \subset R_0 = R_1 = \Big\{ v \in K^\infty : \; T(y)(v) \in -W(y) \quad \forall\, y \in K \Big\}.$$

We introduce three additional cones

$$R \doteq \Big\{ v \in K^\infty : \; T(y)(y + \lambda v - h(y)) \in -P(y) \quad \forall\, y \in K \;\; \forall\, \lambda > 0 \Big\};$$

$$\tilde{R}_1 \doteq \Big\{ v \in K^\infty : \; T(y)(v) \in -W(y) \quad \forall\, y \in K \Big\};$$

$$\tilde{R} \doteq \Big\{ v \in K^\infty : \; T(y)(v) \in -P(y) \quad \forall\, y \in K \Big\}.$$

Remark that $\tilde{R}_1$ is the corresponding cone to the problem (5.3). It follows that $R \subset \tilde{R}$ and, if $E_W \neq \emptyset$ then

$$R \subset \tilde{R} \subset (E_W)^\infty. \tag{5.5}$$

In fact, let us fix any $\bar{x} \in E_W$ and take any $v \in \tilde{R}$. Hypotheses (H0) and (H1) imply, for every $y \in K$,

$$T(y)(\bar{x} + \lambda v - h(y)) = T(y)(\bar{x} - h(y)) + \lambda T(y)(v) \in -W(y) - P(y) \subset \\ \subset -W(y).$$

Hence by Remark 3.3 $T(\bar{x} + \lambda v)(y - \bar{x} - \lambda v) \in W(\bar{x} + \lambda v)$ for all $y \in K$, which says that $\bar{x} + \lambda v \in E_W$ for all $\lambda > 0$. Hence $v \in (E_W)^\infty$. A similar reasoning proves $\tilde{R} \subset R'_0$. Thus, we have the following chain of inclusions

$$R \subset \tilde{R} \subset R'_0 \subset R_0 = R_1 = \tilde{R}_1.$$

If (f'_1) is assumed instead of (f_1), using Proposition 3.6, we obtain $R_1 \subset R$. The latter together with (5.4) imply $\tilde{R}_1 = \tilde{R}$. Hence

$$R = R'_0 = R_0 = R_1 = \tilde{R}_1 = \tilde{R}.$$

On the other hand, since in general R_1 is not convex, a reasonable assumption on R_1 in order that $(**)$ be satisfied, is $R_1 \subset -R_1$. This assumption together with (f'_1) (instead of (f_1)) imply property $(**)$ (see Theorem 3.11). Indeed, every $v \in R_1 \subset -R_1$ satisfies $T(y)(y - \lambda v - h(y)) \in -W(y)$ for all $\lambda > 0$ and all $y \in K$. By Proposition 3.6 $T(y - \lambda v)(y - h(y - \lambda v)) \in W(y - \lambda v)$ for all $\lambda > 0$ and all $y \in K$. Assumption (f'_1) implies $T(y)(y - \lambda v - h(y)) \in -P(y)$ for all $\lambda > 0$ and all $y \in K$. Thus, if $x_k \in K$, $\|x_k\| \to +\infty$, $x_k/\|x_k\| \to v$ with $v \in R'_0 \subset R_0 = R_1$, condition $(**)$ will be satisfied by setting $u = x_k - \|x_k\| v$ for k sufficiently large.

We extend the notion of polar cone to our setting in the following manner. We define the "weak polar cone" of $T(K)$, $(T(K))^0_w$, as follows

$$(T(K))^0_w \doteq \Big\{ u \in \mathbb{R}^n : \ T(y)(u) \in -W(y) \ \ \forall\, y \in K \Big\},$$

whereas the "strong polar cone" of $T(K)$ is

$$(T(K))^0_s \doteq \Big\{ u \in \mathbb{R}^n : \ T(y)(u) \in -P(y) \ \ \forall\, y \in K \Big\}.$$

Certainly both notions coincide when $m = 1$. Hence

$$R_0 = R_1 = K^\infty \cap (T(K))^0_w, \quad \tilde{R} = K^\infty \cap (T(K))^0_s.$$

The previous results are summarized in the next theorems.

THEOREM 5.1. *Let K be a closed convex set in $\mathbb{R}^n$; let P satisfy hypothesis* (H0). *Moreover, let $T : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ be a mapping such that the vector-valued function F given by* (5.1) *satisfies hypothesis* (H1). *Then*

(*a*) $(E_W)^\infty \subset R_0 = R_1 = K^\infty \cap (T(K))^0_w$. *In case* $E_W \neq \emptyset$, $R \subset K^\infty \cap (T(K))^0_s \subset (E_W)^\infty$;
(*b*) $K^\infty \cap (T(K))^0_w = \{0\} \Longrightarrow E_W$ *is non-empty and compact* $\Longrightarrow R = K^\infty \cap (T(K))^0_s = \{0\}$.
(*c*) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r : \ T(x)(y - h(x)) \notin W(x) \Longrightarrow E_W$ *is non-empty and compact. Here* $K_r = \{x \in K : \|x\| \leqslant r\} \neq \emptyset$.

REMARK 5.2. As observed earlier, the set $\tilde{R}_1$ is the corresponding cone R_1 to problem (5.3), and because of the equality (5.4), one may study the possible relationship between problems (5.2) and (5.3). First of all, we see that under the assumption $T(x)(x - h(x)) \in P(x) \cap (-P(x))$ for all $x \in K$, we have

$$\forall\, x, y \in K, \ \ T(x)(y - h(x)) \in W(x) \ \ \Longrightarrow \ \ T(y)(x - h(y)) \in -W(y)$$

if and only if

$$\forall\, x, y \in K, \ \ T(x)(y - x) \in W(x) \ \ \Longrightarrow \ \ T(y)(x - y) \in -W(y).$$

Certainly, $F(x, \cdot)$ is $P(x)$-lower semicontinuous if and only if $G(x, \cdot)$ is so, where $F(x, y) = T(x)(y - x)$ and $G(x, y) = T(x)(y - h(x))$. It remains to analyze the assumption (f_3) of hypothesis (H1) for both problems.

REMARK 5.3. We can go further in the case when $P(x) = \mathbb{R}^m_+$, $W(x) = \mathbb{R}^m \setminus -\text{int}\, \mathbb{R}^m_+$. In such a situation we consider problem (5.3). Writing

$$T(x)(v) = (\langle T_1(x), v\rangle, \ldots, \langle T_m(x), v\rangle),$$

where $T_i(x)$ is the ith-row of the matrix $T(x)$, we get

$$\begin{aligned} R_1 &= \bigcap_{y\in K} \bigcup_{i=1}^{m} \Big\{ v \in K^\infty : \langle T_i(y), v\rangle \leqslant 0 \Big\}, \\ R &= \bigcap_{y\in K} \bigcap_{i=1}^{m} \Big\{ v \in K^\infty : \langle T_i(y), v\rangle \leqslant 0 \Big\} = \bigcap_{i=1}^{m} K^\infty \cap (T_i(K))^0. \end{aligned} \tag{5.6}$$

The condition for all $x, y \in K$,

$$T(x)(y - x) \notin -\text{int}\, \mathbb{R}^m_+ \Longrightarrow T(y)(x - y) \notin \text{int}\, \mathbb{R}^m_+$$

is weaker than the condition that each T_i is pseudomonotone in the sense of Karamardian, which means

$$\langle T_i(x), y - x\rangle \geqslant 0 \Longrightarrow \langle T_i(y), x - y\rangle \leqslant 0.$$

Under the latter condition and upper semicontinuity along lines on $f_i(\cdot, y)$ for $f_i(x, y) = \langle T_i(x), y - x\rangle$, it is known (see Crouzeix, 1997; or Flores-Bazán, 2000) that

$$E_i \ \text{ is non} - \text{empty and compact} \ \ \Longleftrightarrow \ \ K^\infty \cap (T_i(K))^0 = \{0\}.$$

Here E_i is the solution set of the problem

$$\text{find } \bar{x} \in K \text{ such that } \langle T_i(\bar{x}), y - \bar{x} \rangle \geqslant 0 \;\; \forall\, y \in K.$$

Hence, $R_1 = \{0\}$ implies the solution set to (5.3), E_W, is a non-empty compact set and, for $i = 1, ..., m$, E_i is also non-empty and compact (see (5.6)). This implies $R = \{0\}$.

The next theorem is the analogue to the previous theorem when assumption (f_1') is assumed instead of (f_1).

THEOREM 5.4. *Let K, P be as in the preceding theorem and let $T : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ be a mapping such that the vector-valued function F given by* (5.1) *satisfies hypothesis* (H1) *with (f_1') instead (f_1). Then*
(a) $R = R_0' = R_0 = R_1 = K^\infty \cap (T(K))_w^0 = K^\infty \cap (T(K))_s^0$;
(b) *in case* $E_W \neq \emptyset$, $(E_W)^\infty = R_0' = R_0 = R_1 = R = K^\infty \cap (T(K))_w^0 = K^\infty \cap (T(K))_s^0$;
(c) $R_1 \subset -R_1 \Longrightarrow E_W \neq \emptyset$;
(d) $K^\infty \cap (T(K))_w^0 = \{0\} \Longleftrightarrow K^\infty \cap (T(K))_s^0 = \{0\} \Longleftrightarrow R_0 = \{0\} \Longleftrightarrow R_1 = \{0\} \Longleftrightarrow R = \{0\} \Longleftrightarrow R_0' = \{0\} \Longleftrightarrow E_W$ *is non-empty and compact* $\Longleftrightarrow \exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r : T(x)(y - h(x)) \notin W(x)$.

Other existence theorems for vector variational inequalities may be found in Ansari (2000), Chen (1992), Giannessi (2000), Konnov and Yao (1997), Daniilidis and Hadjisavvas (1996), Siddiqi et al. (1995), Yang and Goh (1997).

A particular vector variational problem arises when dealing with vector minimization problems (see Yang and Goh, 1997). Take P as a constant cone satisfying hypothesis (H0). Let $G : K \to \mathbb{R}^m$ be a P-convex vector-valued mapping. It is requested to find

$$\bar{x} \in K \text{ such that } G(y) - G(\bar{x}) \notin -\text{int}\, P \text{ for all } y \in K. \tag{5.7}$$

This problem has been studied in Flores-Bazán (1999) under either the P-convexity condition on G or semi-strict quasiconvexity on each component of G in case $P = \mathbb{R}^m_+$. In the latter situation, some characterizations of the non-emptiness and compactness of the solution set to (5.7) has been established in Deng (1998a, b), see also Chen and Craven (1994) . Since the $\mathbb{R}^m_+$-convexity of G amounts to saying that each component g_i, $i = 1, ..., m$, of G is convex, it is not difficult to prove, by assuming that each g_i is differentiable (see Chen and Craven, 1994, for instance), that problem (5.7) is equivalent to find

$$\bar{x} \in K \text{ such that } DG(\bar{x})(y - \bar{x}) \notin -\text{int}\, \mathbb{R}^m_+ \text{ for all } y \in K. \tag{5.8}$$

Here $DG(x)$ stands for the derivative of G at x, which is a matrix of order $m \times n$ given by the partial derivatives $\partial g_i/\partial x_j$. Problem (5.8) is of the form (3.1). Consequently, by Theorems 3.10 and 3.7 (use $F(x, y) = DG(x)(y - x)$), the solution set of (5.7) or (5.8) is nonempty and bounded (compact) if

$$\left\{x \in K : DG(x)(y - x) \notin -\mathrm{int}\, \mathbb{R}^m_+\right\}$$

is bounded for some $y \in K$. Such a result is the one established in Theorem 2.3 of Chen and Craven (1994). Notice that $(DG(x) - DG(y))(x - y) \in \mathbb{R}^m_+$ for all $x, y \in K$.

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